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On the Solution of the Differential Equation of Sources.

By J. Hammond.

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1. Let

$$c_{2m+1} = a_0^2 a_{2m+1} - u_1 a_0 a_1 a_{2m} + a_{2m-1}(u_2 a_0 a_2 + v_2 a_1^2) - a_{2m-2}(u_3 a_0 a_3 + v_3 a_1 a_2) + \dots \quad (1)$$

where the general term is $a_{2m+1-\lambda}(u_\lambda a_0 a_\lambda + v_\lambda a_1 a_{\lambda-1})$,
and the last term is $\{a_{m+1}(u_m a_0 a_m + v_m a_1 a_{m-1}) - w_m a_1 a_m^2\}$,
be a particular solution of the differential equation

$$\left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \dots + na_{n-1} \frac{d}{da_n}\right) z = 0. \quad (2)$$

Then the coefficients of (1) are found from the equations

$$\left.\begin{aligned}
&2m + 1 = u_1 \\
&2mu_1 = 2u_2 + 2v_2 \\
&(2m - 1)u_2 = 3u_3 + v_3 \\
&(2m - 2)u_3 = 4u_4 + v_4 \\
&\dots\dots\dots\dots\dots \\
&(2m + 1 - \lambda)u_\lambda = (\lambda + 1)u_{\lambda+1} + v_{\lambda+1} \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&(m + 2)u_{m-1} = mu_m + v_m \\
&(m + 1)u_m = w_m
\end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned}
&(2m - 1)v_2 = 2v_3 \\
&(2m - 2)v_3 = 3v_4 \\
&\dots\dots\dots\dots \\
&(2m + 1 - \lambda)v_\lambda = \lambda v_{\lambda+1} \\
&\dots\dots\dots\dots\dots \\
&(m + 2)v_{m-1} = (m - 1)v_m \\
&(m + 1)v_m = 2mw_m
\end{aligned}\right\} \quad (4)$$

The general solution of (4) is

$$v_\lambda = \frac{(2m + 2 - \lambda)(2m + 3 - \lambda) \dots 2m}{(\lambda - 1)!} c \quad (5)$$

where c is independent of λ.

Assume in (3)

$$u_\lambda = \frac{(2m + 2 - \lambda)(2m + 3 - \lambda) \dots (2m + 1)}{\lambda!} \alpha_\lambda$$

then
$$(2m+1)\alpha_\lambda = (2m+1)\alpha_{\lambda+1} + c$$
or
$$\alpha_\lambda = c' - \frac{c\lambda}{2m+1}.$$
To determine c and c' we have, since
$$u_1 = 2m+1\,,\ u_2 + v_2 = \frac{(2m+1)2m}{1.2} = 2mc + \frac{2m(2m+1)}{1.2}\left(c' - \frac{2c}{2m+1}\right)$$
$c' = 1$.

The other constant c must be determined by means of the equations
$$(m+1)u_m = w_m\,,\ (m+1)v_m = 2mw_m\,,\ \text{which give } v_m = 2mu_m$$
or
$$\frac{(m+2)\ldots 2m}{(m-1)!}c = 2m\left(1 - \frac{cm}{2m+1}\right)\frac{(m+2)\ldots(2m+1)}{m!}$$
reducing to
$$c = 2(2m+1-cm) \text{ or } c = 2.$$

Hence the coefficients of c_{2m+1} are

$$u_1 = 2m+1$$
$$u_2 = \frac{(2m+1)2m}{1.2}\left(1 - \frac{4}{2m+1}\right)$$
$$u_3 = \frac{(2m+1)2m(2m-1)}{1.2.3}\left(1 - \frac{6}{2m+1}\right)$$
$$u_4 = \frac{(2m+1)2m(2m-1)(2m-2)}{1.2.3.4}\left(1 - \frac{8}{2m+1}\right)$$
. . . .

$$v_2 = 2.2m$$
$$v_3 = 2.\frac{2m(2m-1)}{1.2}$$
$$v_4 = 2.\frac{2m(2m-1)(2m-2)}{1.2.3.}$$
. . . .
$$w_m = \frac{2m(2m-1)\ldots(m+1)}{m!}.$$

Now since $v_2, v_3, v_4, \ldots v_m$ are double the coefficients of $x, x^2, x^3, \ldots x^{m-1}$ respectively in $(1+x)^{2m}$, and $w_m =$ the coefficient of the middle term, if Q_{2m} denote the quadrinvariant of the $2m^{\text{ic}}$, we have
$$C_{2m+1} = a_0 V_{2m+1} - 2a_1 Q_{2m} \tag{6}$$
where V_{2m+1} satisfies a similar differential equation to (2).

For if we call the operator of (2) δ, we have $\delta C_{2m+1} = 0$, $\delta Q_{2m} = 0$, and the result of operating on (6) with δ is
$$\delta V_{2m+1} = 2Q_{2m}.$$

2. The system of simultaneous equations
$$\frac{dz}{0} = \frac{da_1}{a_0} = \frac{da_2}{2a_1} = \frac{da_3}{3a_2} = \ldots \frac{da_n}{na_{n-1}} \tag{7}$$
are satisfied by $z =$ const.
$$H = a_0 a_2 - a_1^2 = \text{const.}$$
$$C_3 = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3 = \text{const.}$$
. . . .
$$Q_{2m} = \text{const.},\ C_{2m+1} = \text{const.}, \ldots$$

For if we multiply both numerator and denominator of the first two fractions by $\frac{dH}{da_1}$, $\frac{dH}{da_2}$ respectively and add, each fraction $= \frac{dH}{0}$; the first three fractions giving in like manner $\frac{dC_3}{0}$; and so on for Q_{2m}, C_{2m+1}, . . .

Thus the general solution of (2) is

$$z = \phi(H,\ C_3,\ Q_4,\ C_5,\ \ldots) \tag{8}$$

the last letter being Q_n or C_n according as n is even or odd.

In proceeding from (7) to (8) any other set of $n-1$ particular solutions of (2) might have been used instead of H, C_3, Q_4, C_5, . . .; and the result would have been a perfectly general solution, provided all the letters $a_0, a_1, a_2, a_3, \ldots a_n$ were involved in the set chosen. But since H, C_3, Q_4, C_5, . . . are ground-sources, and ground-sources too of lower degree than any other set, (8) is the simplest possible form of the general solution of (2).

For all weights, up to 10 inclusive, we have

$$\begin{aligned}
U &= a_0 \\
H &= a_0 a_2 - a_1^2 \\
C_3 &= a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3 \\
Q_4 &= a_0 a_4 - 4a_1 a_3 + 3a_2^2 \\
C_5 &= a_0^2 a_5 - 5a_0 a_1 a_4 + 2a_0 a_2 a_3 + 8a_1^2 a_3 - 6a_1 a_2^2 \\
Q_6 &= a_0 a_6 - 6a_1 a_5 + 15a_2 a_4 - 10a_3^2 \\
C_7 &= a_0^2 a_7 - 7a_0 a_1 a_6 + 9a_0 a_2 a_5 + 12a_1^2 a_5 - 5a_0 a_3 a_4 - 30a_1 a_2 a_4 + 20a_1 a_3^2 \\
Q_8 &= a_0 a_8 - 8a_1 a_7 + 28a_2 a_6 - 56a_3 a_5 + 35a_4^2 \\
C_9 &= a_0^2 a_9 - 9a_0 a_1 a_8 + 20a_0 a_2 a_7 + 16a_1^2 a_7 - 28a_0 a_3 a_6 - 56a_1 a_2 a_6 \\
&\qquad + 14a_0 a_4 a_5 + 112a_1 a_3 a_5 - 70a_1 a_4^2 \\
Q_{10} &= a_0 a_{10} - 10a_1 a_9 + 45a_2 a_8 - 120a_3 a_7 + 210a_4 a_6 - 126a_5^2,
\end{aligned}$$

where U is the source of the Quantic $(a_0, a_1, a_2, a_3, \ldots a_n)(x, y)^n$, H that of its Hessian, C_3 a ground-source for the Cubic and all higher Quantics, Q_4 for the Quartic and all higher Quantics; and generally Q_{2m} is a ground-source (the Quadrinvariant) for the $2m^{\text{ic}}$ and for all higher Quantics, the source of a covariant, C_{2m+1} is a ground-source for the $(2m+1)^{\text{ic}}$ and all higher Quantics.

3. A glance at the first terms of U, H, C_3, Q_4, C_5, . . . as written above, will suffice to show that none of the powers or products of these quantities is a linear function of any of the rest, and that the number of linearly independent compounds, of degree = weight $= w$, that can be formed of them is equal to the number of terms, of degree = weight $= w$, that can be formed of the letters $a_0, a_2, a_3, \ldots a_n$ alone. But these terms are equal in number to the linearly

independent sources of degree = weight = w; for the general function of degree = weight = w may be formed by multiplying the general function of degree = weight = $w-1$ by a_1 and adding on those terms that can be formed, of degree = weight = w, with all the letters except a_1.

Hence all the sources of degree = weight = w are linear functions of the compounds of U, H, C_3, Q_4, C_5, ... ; and if $w > 3$ there can be no ground-source whose degree and weight are equal.

Or if S denote a source of degree μ and weight w,

$$SU^{w-\mu} = \phi(U,\ H,\ C_3,\ Q_4,\ C_5,\ \ldots) \tag{9}$$

where ϕ is rational, integral, and homogeneous, of degree = weight = w.

Thus in the case $w = 9$,

$$\begin{aligned} SU^{9-\mu} = \alpha_1 H^3 C_3 + \alpha_2 C_3^3 + U^2(\alpha_3 H C_3 Q_4 + \alpha_4 H^2 C_5) \\ + U_4(\alpha_5 Q_4 C_5 + \alpha_6 C_3 Q_6 + \alpha_7 H C_7) + \alpha_8 U^6 C_9 \end{aligned} \tag{10}$$

where α_1, α_2, α_3, ... are arbitrary constants.

When the order of the Quantic (n) is not less than w, the last suffix in (9) is w; but if $n < w$, every letter whose suffix is greater than n must be struck out. Thus in the case of the Octavic the term containing C_9 disappears from (10), and in the case of the Quartic (10) becomes $SU^{9-\mu} = (\alpha_1 H^3 + \alpha_2 C_3^2 + \alpha_3 U^2 H Q_4) C_3$, and every source of weight 9 for either the Quartic or Cubic contains C_3 as a factor.

When the arbitrary constants that occur in (9) are so chosen that S, or one or more of its factors, is a ground-source distinct from H, C_3, Q_4, C_5, ... the result is a syzygant. The total number of such syzygants is the total number of ground-sources of weight w + the total number of compounds formed by multiplying together ground-sources of inferior weight, rejecting those that are made up of H, C_3, Q_4, C_5, ... only; or it is the total number of compounds of degree = weight = w minus the number of linearly independent sources. In all cases where syzygants of other forms occur they are deducible from syzygants of this form and the same weight. No syzygant of weight $2m$ contains either Q_{2m} or C_{2m-1}, and no syzygant of weight $2m+1$ contains either C_{2m+1} or Q_{2m}.

The first example of a syzygant occurs when $w = 6$, in which case

$$TU^3 = U^2 H Q_4 - 4H^3 - C_3^2 \tag{11}$$

where T is the cubinvariant of the Quartic and a ground-source for all higher Quantics. The corresponding syzygant for the Cubic, viz.

$$U^2 \Delta - 4H^3 - C_3^2 = 0$$

is deduced from this as follows: we retain only the last two terms of (11), as being the only terms corresponding to the Cubic, and noticing that the rejected terms are divisible by U^2 we replace them by a single term $U^2\Delta$. Hence Δ, the Discriminant of the Cubic is a *Special Form; i. e.* it is a groundform for the Cubic, but not for any higher Quantic. By this method we are enabled to detect Special Forms whenever they exist.

When $w = 7$ there is only one syzygant, viz. after reduction by dividing out a factor U^2, we have

$$UP = HC_5 - C_3 Q_4 \tag{12}$$

where P is of degree 4, and a ground-source for the Quintic and all higher Quantics. There are obviously no Special Forms of weight 7. For the Quintic and Sextic there is only one ground-source of weight 7, viz. P (source of the covariant 4.6 for the Quintic, 4.10 for the Sextic, and $4.4n-14$ for the n^{ic}); the Septimic and all higher Quantics have also the ground-source C_7 of weight 7, (source of $3.3n-14$ for the n^{ic}). It may also be noticed that the order of the groundform P is higher than that of any other groundform of the fourth degree, and that H and C_3 are the highest ordered quadratic and cubic forms respectively; for example H, C_3 and P are the sources of 2.16, 3.24, and 4.26 respectively for the Decimic.

4. Those syzygants in which S is a compound are formed immediately from the syzygants of inferior weight. Thus in the case $w = 9$ the only compounds, formed by multiplying together ground-sources of inferior weight, are, besides the powers and products of H, C_3, Q_4, ... $C_3 T$ and HP; and from (11) and (12) we obtain immediately

$$\left.\begin{aligned} C_3 T U^3 &= U^2 H C_3 Q_4 - 4H^3 C_3 - C_3^3 \\ HPU &= H^2 C_5 - HC_3 Q_4 \end{aligned}\right\} \tag{13}$$

To find the other syzygants we must determine the arbitrary constants in (10) so that S may be a ground-source, distinct from C_9. Hence $\alpha_8 = 0$ and

$$SU^{9-\mu} = \alpha_1 H^3 C_3 + \alpha_2 C_3^3 + U^2(\alpha_3 H C_3 Q_4 + \alpha_4 H^2 C_5) + U^4(\alpha_5 Q_4 C_5 + \alpha_6 C_3 Q_6 + \alpha_7 H C_7).$$

Now since S is a ground-source $\mu < 9$, and the above relation can only be identical when both sides are divisible by U, which can only happen when $\alpha_1 = 4\alpha_2$ and

$$SU^{9-\mu} = \alpha_2(4H^3 + C_3^2)\, C_3 + U^2(\alpha_3 H C_3 Q_4 + \alpha_4 H^2 C_5) + U^4(\alpha_5 Q_4 C_5 + \alpha_6 C_3 Q_6 + \alpha_7 H C_7).$$

The right-hand side of this when divided by U^2 consists of compound terms, so that $\mu < 7$ and both sides must be divisible by U^3 if S is to be a ground-source.

For this it is necessary that $\alpha_2 + \alpha_3 + \alpha_4 = 0$, whence

$$\begin{aligned} SU^{9-\mu} = \alpha_2(4H^3 + C_3^2 - U^2 H Q_4)\, C_3 + \alpha_4 U^2 H(HC_5 - C_3 Q_4) \\ + U^4(\alpha_5 Q_4 C_5 + \alpha_6 C_3 Q_6 + \alpha_7 H C_7). \end{aligned}$$

Or reducing by means of (13)

$$SU^{6-\mu} = \alpha_4 HP - \alpha_2 C_3 T + U(\alpha_5 Q_4 C_5 + \alpha_6 C_3 Q_6 + \alpha_7 HC_7) \qquad (14)$$

With the values

$$T = a_0(a_2 a_4 - a_3^2) - (a_1^2 a_4 - 2a_1 a_2 a_3 + a_2^3)$$
$$P = a_0^2(a_2 a_5 - a_3 a_4) - a_0(a_1^2 a_5 + 2a_1 a_2 a_4 - 4a_1 a_3^2 + a_2^2 a_3) + 3a_1(a_1^2 a_4 - 2a_1 a_2 a_3 + a_2^3)$$

and those of H, C_3, . . . we are able to determine the arbitrary constants of (14) so that the right-hand side is divisible by a_0^2; when

$$\left.\begin{aligned} 3\alpha_4 = 2\alpha_2 = 6\beta \\ \beta = -6\alpha_5 = 6(\alpha_6 + \alpha_7) \end{aligned}\right\}$$

or if $\beta = 6\gamma$, $\alpha_5 = -\gamma$, $\alpha_6 = \gamma - \alpha_7$, (14) becomes

$$SU^{6-\mu} = \gamma\{U(C_3 Q_6 - Q_4 C_5) + 12HP - 18C_3 T\} + \alpha_7 U(HC_7 - C_3 Q_6).$$

Now suppose

$$SU^2 = U(C_3 Q_6 - Q_4 C_5) + 12HP - 18C_3 T$$
$$S'U = HC_7 - C_3 Q_6$$

where by actual calculation

$$S = a_0^2(a_3 a_6 - a_4 a_5) - a_0(3a_1 a_2 a_6 + 2a_1 a_3 a_5 - 9a_2^2 a_5 - 5a_1 a_4^2 + 17a_2 a_3 a_4 - 8a_3^3)$$
$$+ 2a_1(a_1^2 a_6 - 3a_1 a_2 a_5 + a_1 a_3 a_4 + 3a_2^2 a_4 - 2a_2 a_3^2)$$
$$S' = a_0^2(a_2 a_7 - a_3 a_6) - a_0(a_1^2 a_7 + 4a_1 a_2 a_6 - 6a_1 a_3 a_5 - 9a_2^2 a_5 + 20a_2 a_3 a_4 - 10a_3^3)$$
$$+ 5a_1(a_1^2 a_6 - 3a_1 a_2 a_5 + a_1 a_3 a_4 + 3a_2^2 a_4 - 2a_2 a_3^2).$$

Hence

$$2S' - 5S = UR,$$

where

$$R = a_0(2a_2 a_7 - 7a_3 a_6 + 5a_4 a_5)$$
$$- (2a_1^2 a_7 - 7a_1 a_2 a_6 - 22a_1 a_3 a_5 + 27a_2^2 a_5 + 25a_1 a_4^2 - 45a_2 a_3 a_4 + 20a_3^3)$$

And finally

$$U^2 R = 2(HC_7 - C_3 Q_6) - 5US,$$

showing that R and S are ground-sources.

When written in their standard form, the syzygants which give R and S are

$$\left.\begin{aligned} SU^5 &= U^4(C_3 Q_6 - Q_4 C_5) + 6U^2 H(2HC_5 - 5C_3 Q_4) + 18C_3(C_3^2 + 4H^3) \\ RU^6 &= U^4(2HC_7 - 7C_3 Q_6 + 5Q_4 C_5) - 30U^2 H(2HC_5 - 5C_3 Q_4) - 90C_3(C_3^2 + 4H^3) \end{aligned}\right\} (15)$$

and, by the method of the preceding article, either of these will give the same Special Form Λ for the Quintic, which is the source of the covariant 5.7 and is given by the syzygant

$$U^4 \Lambda = U^2 H(2HC_5 - 5C_3 Q_4) + 3C_3(C_3^2 + 4H^5).$$

The second equation of (15) gives no Special Form for the Sextic, for if we write

$U^4(A - 7C_3 Q_6 + 5Q_4 C_5) -$ &c. $= 0$ we have $U^4(A - 2C_3 Q_6) = 5SU^5$, or $A = 2C_3 Q_6 + 5SU$ which is a compound form.

It need hardly be noticed that the value of Λ given above is there given in its simplest form, whereas the rejection of the terms containing S and Q_6 in the first equation of (15) and those containing R, Q_6, and C_7 in the second, would give two sources which though different are not really distinct from Λ.

The complete list of ground-sources of weight 9 is: for the Quintic Λ, source of the covariant 5.7; for the Sextic and all higher Quantics S, source of 4.6 for the Sextic, $4.4n-18$ for the n^{ic}; for the Septimic and all higher Quantics R, source of 3.3 for the Septimic, $3.3n-18$ for the n^{ic}; for the Nonic and all higher Quantics C_9, source of 3.9 for the Nonic, $3.3n-18$ for the n^{ic}. These results agree with Prof. Sylvester's Tables of Groundforms (American Journal, Vol. II, p. 223–251).

The syzygants (15) when reduced to their lowest terms become

$$\left.\begin{aligned} SU^2 &= U(C_3 Q_6 - Q_4 C_5) + 12HP - 18C_3 T \\ RU^2 + 5SU &= 2(HC_7 - C_3 Q_6) \end{aligned}\right\} \qquad (16)$$

of which the first is a ground-syzygant for the Sextic and all higher Quantics, and the second is a ground-syzygant for the Septimic and all higher Quantics. These with the special ground-syzygant for the Quintic

$$U\Lambda = 2HP - 3C_3 T$$

form a complete set of ground-syzygants of weight 9 for all Quantics.

5. If in equation (6) we write for an instant $a_0 = 0$ we have

$$\begin{aligned} C_{2m+1} &= -2a_1 Q_{2m} \\ C_{2p+1} &= -2a_1 Q_{2p} \\ C_{2q+1} &= -2a_1 Q_{2q} \\ &\cdots\cdots\cdots \end{aligned}$$

whence it is easy to deduce

$$\begin{aligned} &C^2_{2m+1} + 4HQ^2_{2m} = U(*) \\ &C_{2m+1} Q_{2p} - C_{2p+1} Q_{2m} = U(*) \\ &C_{2m+1} C_{2p+1} + 4HQ_{2m} Q_{2p} = U(*) \\ &C^3_{2m+1} + 4C_3 Q^3_{2m} = U(*) \\ &C^2_{2m+1} Q_{2p} Q_{2q} - Q^2_{2m} C_{2p+1} C_{2q+1} = U(*) \\ &C_{2m+1} C_{2p+1} C_{2q+1} + 4C_3 Q_{2m} Q_{2p} Q_{2q} = U(*) \end{aligned}$$

with many other relations of like nature.

The expressions on the left-hand are of frequent occurrence in syzygants; the particular case $m=1$ gives $C_3^2 + 4H^3 = U^2\Delta$, for weight 8 there is a syzygant of the form $C_3 C_5 + 4H^2 Q_4 = U(*)$, and for weight 10 one of the form $C_5^2 + 4HQ_4^2 = U(*)$; other examples are easily found.

In the actual calculation of ground-sources equation (6) is sometimes useful; thus the relations $C_{2m+1} = a_0 V_{2m+1} - 2a_1 Q_{2m}$ and $C_{2p+1} = a_0 V_{2p+1} - 2a_1 Q_{2p}$ give

$$C_{2m+1} Q_{2p} - C_{2p+1} Q_{2m} = U(V_{2m+1} Q_{2p} - V_{2p+1} Q_{2m}),$$

and the advantage consists in this that V_{2m+1} has only $m+1$ terms, whilst C_{2m+1} has $2m+1$ terms.

Thus in calculating the ground-source P of weight 7 from the syzygant

$$PU = HC_5 - C_3 Q_4$$

we have

$$P = HV_5 - V_3 Q_4 = (a_0a_2 - a_1^2)(a_0a_5 - 3a_1a_4 + 2a_2a_3) - (a_0a_3 - a_1a_2)(a_0a_4 - 4a_1a_3 + 3a_2^2)$$

the expanded value is given already.

It may be noticed that Q_{2m} and V_{2m+1} may be expressed as the sum of determinants of the form $(a_p a_q - a_{p+1} a_{q-1})$; thus

$$H = a_0 a_2 - a_1^2$$
$$Q_4 = (a_0 a_4 - a_1 a_3) - 3(a_1 a_3 - a_2^2)$$
$$Q_6 = (a_0 a_6 - a_1 a_5) - 5(a_1 a_5 - a_2 a_4) + 10(a_2 a_4 - a_3^2)$$
$$Q_8 = (a_0 a_8 - a_1 a_7) - 7(a_1 a_7 - a_2 a_6) + 21(a_2 a_6 - a_3 a_5) - 35(a_3 a_5 - a_4^2)$$
$$Q_{10} = (a_0 a_{10} - a_1 a_9) - 8(a_1 a_9 - a_2 a_8) + 36(a_2 a_8 - a_3 a_7) - 84(a_3 a_7 - a_4 a_6) + 126(a_4 a_6 - a_5^2)$$

. . . .

the coefficients of the determinants in Q_{2m} being those of the binomial expansion of $(1+x)^{2m-1}$.

To express V_{2m+1} in this way, observe that if $(a_p a_q - a_{p+1} a_{q-1}) = (p, q)$

$$\delta(p, q) = \left\{ p a_{p-1} \frac{d}{da_p} + q a_{q-1} \frac{d}{da_q} + (p+1) a_p \frac{d}{da_{p+1}} + (q-1) a_{q-2} \frac{d}{da_{q-1}} \right\} (p, q) \cdot$$
$$= p(p-1, q) + (q-1)(p, q-1)$$

and the coefficients of V_{2m+1} are found from the differential equation

$$\delta V_{2m+1} = 2 Q_{2m} \, .$$

The values obtained are

$$V_3 = a_0 a_3 - a_1 a_2$$
$$V_5 = (a_0 a_5 - a_1 a_4) - 2(a_1 a_4 - a_2 a_3)$$
$$V_7 = (a_0 a_7 - a_1 a_6) - 4(a_1 a_6 - a_2 a_5) + 5(a_2 a_5 - a_3 a_4)$$
$$V_9 = (a_0 a_9 - a_1 a_8) - 6(a_1 a_8 - a_2 a_7) + 14(a_2 a_7 - a_3 a_6) - 14(a_3 a_6 - a_4 a_5)$$

. . . .

and the coefficients are not Binomial Coefficients.

DISPROOF OF PROF. SYLVESTER'S FUNDAMENTAL POSTULATE.

The development of the G. F. for the Binary Septimic is

$$1 + ax^7 + a^2(x^2 + x^6 + x^{10} + x^{14}) + a^3(x^3 + x^5 + x^7 + 2x^9 + x^{11} + \ldots)$$
$$+ a^4(\ldots + x^6 + \ldots) + a^5(\ldots + 4x^{13} + \ldots) + \ldots$$

whence we see that there are four linearly independent covariants of deg-order (5.13), and exactly 4 compound covariants of the same deg-order, viz.

(1.7)(4.6), (2.2)(3.11), (2.6)(3.7), (2.10)(3.3).

Or with the notation of the present paper, using F to denote the covariant (4.6)

$$UF, \quad Q_6 C_5, \quad Q_4 C_7, \quad HR.$$

If then the Fundamental Postulate is universally true, the Binary Septimic has neither a groundform nor a syzygant of deg-order (5.13).

But among other relations it has been proved in the preceding article that

$$C_{2m+1}Q_{2p}-C_{2p+1}Q_{2m}=U(*)\,;$$

where obviously $(*)$ is a source of degree 4 and weight $2m+2p+1$, or what is the same thing a covariant of deg-order $(4.4n-4m-4p-2)$ for the n^{ic}.

In particular if $n=7$, $m=3$, $p=2$; $(*)$ is of deg-order (4.6), and since there is only one such covariant, viz. F, we have

$$Q_4C_7-C_5Q_6=UF.$$

Thus the number of linearly independent compounds is reduced to 3, and a groundform of deg-order (5.13) is necessary to make up the required number of independent forms, *i. e.* the Binary Septimic has both a groundform and a syzygant of deg-order (5.13).

Let Θ denote the source of the new groundform (5.13), then the values of F and Θ are found to be

$$\begin{aligned}
F= {}& a_0^2(a_4a_7-a_5a_6)\\
&-a_0(4a_1a_3a_7-3a_2^2a_7+2a_1a_4a_6+2a_2a_3a_6-6a_1a_5^2+6a_2a_4a_5-10a_3^2a_5+5a_3a_4^2)\\
&+20a_1^2a_3a_6-15a_1a_2^2a_6-18a_1^2a_4a_5-24a_1a_2a_3a_5+27a_2^3a_5+45a_1a_2a_4^2\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad-10a_1a_3^2a_4-45a_2^2a_3a_4+20a_2a_3^3\\
\Theta= {}& a_0^2(a_2a_4a_5-3a_3^2a_5+2a_3a_4^2)\\
&-a_0(a_1^2a_4a_5-14a_1a_2a_3a_5+9a_2^3a_5+11a_1a_2a_4^2+a_1a_3^2a_4-14a_2^2a_3a_4+6a_2a_3^3)\\
&-a_1(8a_1^2a_3a_5-6a_1a_2^2a_5-9a_1^2a_4^2+16a_1a_2a_3a_4-3a_2^3a_4+2a_2^2a_3^2-8a_1a_3^3).
\end{aligned}$$

The compound sources of the same degree-order are

$$\begin{aligned}
UF= {}& a_0^3(a_4a_7-a_5a_6)-a_0^2(4a_1a_3a_7-3a_2^2a_7+\text{\&c.})+a_0(20a_1^2a_3a_6\ldots)\\
HR= {}& a_0^2(2a_2^2a_7-7a_2a_3a_6+5a_2a_4a_5)\\
&-a_0(4a_1^2a_2a_7-7a_1^2a_3a_6-7a_1a_2^2a_6+5a_1^2a_4a_5-22a_1a_2a_3a_5+27a_2^3a_5\\
&\qquad\qquad\qquad\qquad\qquad\qquad+25a_1a_2a_4^2-45a_2^2a_3a_4+20a_2a_3^3)\\
&+a_1^2(2a_1^2a_7-7a_1a_2a_6-22a_1a_3a_5+27a_2^2a_5+25a_1a_4^2-45a_2a_3a_4+20a_3^3)\\
C_7Q_4= {}& a_0^3a_4a_7-a_0^2(4a_1a_3a_7-3a_2^2a_7+7a_1a_4a_6-9a_2a_4a_5+5a_3a_4^2)\\
&+a_0(28a_1^2a_3a_6-21a_1a_2^2a_6-36a_1a_2a_3a_5+12a_1^2a_4a_5+27a_2^3a_5-30a_1a_2a_4^2\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad+40a_1a_3^2a_4-15a_2^2a_3a_4)\\
&-48a_1^3a_3a_5+36a_1^2a_2^2a_5+120a_1^2a_2a_3a_4-90a_1a_2^3a_4-80a_1^2a_3^3+60a_1a_2^2a_3^2\\
C_5Q_6= {}& a_0^3a_5a_6-a_0^2(5a_1a_4a_6-2a_2a_3a_6+6a_1a_5^2-15a_2a_4a_5+10a_3^2a_5)\\
&+a_0(8a_1^2a_3a_6-6a_1a_2^2a_6+30a_1^2a_4a_5-12a_1a_2a_3a_5-75a_1a_2a_4^2+30a_2^2a_3a_4\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad+50a_1a_3^2a_4-20a_2a_3^3)\\
&-48a_1^3a_3a_5+36a_1^2a_2^2a_5+120a_1^2a_2a_3a_4-90a_1a_2^3a_4-80a_1^2a_3^3+60a_1a_2^2a_3^2.
\end{aligned}$$

Whence it is easy to verify the relation

$$C_7Q_4-C_5Q_6=UF$$

and to see that Θ, HR, $C_7 Q_4$, $C_5 Q_6$, are not connected by any syzygy, or that Θ which is a ground-source for the Quintic and Sextic is also a ground-source for the Septimic.

SPECIMEN TABLE OF GROUNDFORMS FOR ALL BINARY QUANTICS.

2	3	4	5	6	7	8	9	10	n	Weight	Name	SYZYGANTS.
1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	1.10	$1.n$	0	U	
$\boxed{2.0}$	2.2	2.4	2.6	2.8	2.10	2.12	2.14	2.16	$2.2n-4$	2	H	
	3.3	3.6	3.9	3.12	3.15	3.18	3.21	3.24	$3.3n-6$	3	C_3	
		$\boxed{2.0}$	2.2	2.4	2.6	2.8	2.10	2.12	$2.2n-8$	4	Q_4	
			3.5	3.8	3.11	3.14	3.17	3.20	$3.3n-10$	5	C_5	
	$\boxed{4.0}$									6	Δ	$U^2\Delta = C_3^2 + 4H^3$
		$\boxed{3.0}$	3.3	3.6	3.9	3.12	3.15	3.18	$3.3n-12$	6	T	$U^3 T = U^2 HQ_4 - C_3^2 - 4H^3$
				$\boxed{2.0}$	2.2	2.4	2.6	2.8	$2.2n-12$	6	Q_6	
			4.6	4.10	4.14	4.18	4.22	4.26	$4.4n-14$	7	P	$UP = HC_5 - C_3 Q_4$
					3.7	3.10	3.13	3.16	$3.3n-14$	7	C_7	
			4.4							8		
				3.2	3.5	3.8	3.11	3.14	$3.3n-16$	8		
						$\boxed{2.0}$	2.2	2.4	$2.2n-16$	8	Q_8	
			5.7							9	Λ	$U\Lambda = 2HP - 3C_3 T$
				4.6	4.10	4.14	4.18	4.22	$4.4n-18$	9	S	$U^2 S = U(C_3 Q_6 - Q_4 C_5) + 12HP - 18C_3 T$
					3.3	3.6	3.9	3.12	$3.3n-18$	9	R	$U^2 R + 5US = 2(HC_7 - C_3 Q_6)$
							3.9	3.12	$3.3n-18$	9	C_9	
			$\boxed{4.0}$	4.4	4.8	4.12	4.16	4.20	$4.4n-20$	10		
					4.8					10		
						3.4	3.7	3.10	$3.3n-20$	10		
								$\boxed{2.0}$	$2.2n-20$	10	Q_{10}	

The arrangement of this table is by the weight of sources, given in the weight column; the table gives a complete list of groundforms of all weights up to 10, those with names and the syzygants are given in the present paper, the rest are copied from Prof. Sylvester's tables.

To embrace all Prof. Sylvester's results it would be necessary to continue the above table to weight 85.

Invariants are enclosed in dark lines to catch the eye.